VII. *On the Equilibrium of Fluids, and the Figure of a Homogeneous Planet in a Fluid State. By* James Ivory, *A.M. F.R.S. Instit. Reg. Sc. Paris. Corresp. et Reg. Sc. Gottin. Corresp.*

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I. *Equilibrium of Fluids.*

1. THE nature of the ultimate particles of a fluid, and the peculiar manner of their mutual connection, are entirely unknown to us. We conceive that they obey the same mechanical laws to which all matter is subject. Experience shows that the particles of a fluid move freely among one another, yielding to the least pressure in any direction; and this is the most general property of such bodies that has yet been discovered. The perfect mobility of their particles must therefore, in the present state of our knowledge, be considered as constituting the definition of fluid bodies, and as the foundation of all our reasoning concerning them. We here confine our attention to a fluid in equilibrium, or at rest, in which state every particle is pressed equally on all sides. It is evident that the mobility of the particles among one another, and their readiness to obey any new impulse, is nowise impeded by the magnitude of their mutual pressure, since this acts at every point with the same intensity in all directions.

If we set aside the effect of gravity, and of all accelerating forces, it follows, from the definition, that the pressure will be equal in all parts of a continuous fluid at rest. In this state we must conceive that the particles are equally distant, and arranged similarly about every interior point. Their mutual distance, it is natural to think, must be connected with the magnitude of pressure; so that when they are more pressed, they will approach one another, and the volume will be diminished; and, when they are less pressed, they will recede from one another, and the volume will be enlarged. Accordingly it is found that no fluid is perfectly incompressible. But in some, such as water

and other liquids, a very great external force must be applied to produce an almost imperceptible variation of bulk; while in others, such as air and the gases, very notable changes of volume are caused by moderate compression. In the investigation of the properties of the first sort of fluids, to which our attention is here exclusively directed, we shall throw out of view the very small degree of compressibility they possess, and shall suppose them to retain the same bulk whatever changes of figure or pressure they may undergo.

In a fluid in equilibrium, the action of the accelerating forces that urge the particles must be counterbalanced by the pressure propagated through the mass: to find the relation between these opposite forces must therefore be the first object of research.

2. Assuming three planes intersecting at right angles which, by the co-ordinates drawn to them, ascertain the position of the particles of the fluid, we shall suppose two points or particles (x, y, z) and $(x + \delta x,\ y + \delta y,\ z + \delta z)$ at the infinitely small distance δs from one another; and we shall put ω for the small base of an upright cylinder or prism of the fluid placed between the two points, and having δs for its length: then the density of the fluid being invariable and represented by unit, and the quantity of matter of the cylinder or prism being denoted by dm, we shall have

$$dm = \omega \times \delta s.$$

Let all the accelerating forces which act upon the particle (x, y, z) be reduced to the directions of the coordinates; and put X, Y, Z for the sums of the reduced forces respectively parallel to x, y, z; then because $\frac{\delta x}{\delta s}, \frac{\delta y}{\delta s}, \frac{\delta z}{\delta s}$ are the cosines of the angles which the line δs makes with x, y, z, the partial forces urging the particle in the direction of δs, will be $\mathrm{X}\frac{\delta x}{\delta s}$, $\mathrm{Y}\frac{\delta y}{\delta s}$, $\mathrm{Z}\frac{\delta z}{\delta s}$, and, if we put

$$f = \mathrm{X}\frac{\delta x}{\delta s} + \mathrm{Y}\frac{\delta y}{\delta s} + \mathrm{Z}\frac{\delta z}{\delta s},$$

the whole accelerating force urging the particle (x, y, z) in the direction of δs, will be equal to f. Multiply now by the equal quantities dm and $\omega\delta s$, and the result will be

$$f\,dm = \omega\,(\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z).$$

As the quantities δx, δy, δz, δs, ω may be assumed as small as we please, the force f may be considered as retaining the same value for all the particles of the cylinder or prism; and therefore $f\,dm$ is the motive force of the cylinder or prism, or the effort it makes to move in the direction of δs from the point (x, y, z) to the point $(x+\delta x, y+\delta y, z+\delta z)$.

Let p represent the hydrostatic pressure of the fluid at the point (x, y, z). This term is used to denote the pressure relatively to the surface pressed: it is the whole pressure any surface sustains divided by the extent of surface; or it is the actual pressure reduced to the unit of surface. The hydrostatic pressure is obviously variable in the different parts of a fluid, the particles of which are urged by accelerating forces; and as it can vary only when its point of action is changed, it must be a function of the coordinates of that point. The whole pressure upon the end of the cylinder or prism at the point (x, y, z) will be equal to $p \times \omega$; for we may suppose that p undergoes no change in the small extent of the surface ω: and, in like manner, the whole pressure upon the opposite end will be equal to $(p+\delta p) \times \omega$. As the pressures upon the two ends act against one another, their effect to move the cylinder or prism in the direction of δs from the point $(x+\delta x, y+\delta y, z+\delta z)$ to the point (x, y, z) will be equal to $\delta p \times \omega$; and this force, on the supposition that the particles of the fluid are at rest, must be equal to $f\,dm$, the directly opposite effect caused by the accelerating forces. We therefore have this equation for expressing the non-effect of the equal and opposite forces, viz.

$$\delta p \times \omega + f\,dm = 0:$$

and, if we substitute the value of $f\,dm$ found before, we shall get

$$\delta p + X\,\delta x + Y\,\delta y + Z\,\delta z = 0. \qquad (1)$$

This equation must take place at every point of the mass of fluid without any relation being supposed between the variations δx, δy, δz; which condition will not be fulfilled unless p be a function of the three independent variables x, y, z. We therefore have

$$\delta p = \frac{dp}{dx}\,\delta x + \frac{dp}{dy}\,\delta y + \frac{dp}{dz}\,\delta z:$$

and, if we substitute this value of δp in the formula (1), the independence of the variations will require these three separate equations,

$$\frac{dp}{dx} = -\mathrm{X}$$

$$\frac{dp}{dy} = -\mathrm{Y}$$

$$\frac{dp}{dz} = -\mathrm{Z}.$$

From this it appears that the algebraic expressions of the forces are not entirely arbitrary; for they must be equal to the partial differential coefficients of a function of three independent variables. By differentiating we shall readily obtain the following equations which do not contain the function p, viz.

$$\frac{d\mathrm{X}}{dy} = \frac{d\mathrm{Y}}{dx}, \quad \frac{d\mathrm{X}}{dz} = \frac{d\mathrm{Z}}{dx}, \quad \frac{d\mathrm{Y}}{dz} = \frac{d\mathrm{Z}}{dy}.$$

Unless the forces possess these properties, which are the well-known conditions of integrability, the equation (1) will not hold in all parts of the mass of fluid, and the equilibrium will be impossible. But in the physical questions that actually occur, the forces of nature being either attractions or repulsions directed to fixt centres, and proportional to certain functions of the distances from those centres, they necessarily fulfil the conditions of integrability.

The whole of what has been said is succinctly expressed by the two following equations,

$$\left.\begin{array}{l} \varphi = \int (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz), \\ p = \mathrm{C} - \varphi, \end{array}\right\} \qquad (2)$$

Here φ represents a function of three independent variables x, y, z without any arbitrary quantity; the constant C required by the integration is necessary only in the expression of p.

3. The hydrostatic pressure at every point of the mass of fluid in equilibrium, is expressed by the second of the equations (2), viz.

$$p = \mathrm{C} - \varphi.$$

But at all those parts of the outer surface of the fluid which are unconfined and entirely at liberty, there is no pressure; wherefore we have, for the equation of all such surfaces,

$$\varphi = \mathrm{C}.$$

It may be proper to remark, that although this equation is universally true, yet it is no new or independent condition of the equilibrium; it is merely an inference from the general expression of the hydrostatic pressure.

If we assume two points (x, y, z) and $(x + dx, y + dy, z + dz)$ indefinitely near one another in a part of the outer surface at liberty, we shall have, in consequence of the foregoing equation,

$$\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy + \frac{d\phi}{dz}dz = 0;$$

or, which is the same thing,

$$X\,dx + Y\,dy + Z\,dz = 0;$$

and if ds represent the distance of the two points, we obtain

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = 0.$$

Now $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the cosines of the angles which the directions of the forces make with the line ds; wherefore the expression on the left side of the foregoing formula is the sum of the partial forces which act in the direction of ds; and as this sum is equal to zero in all positions of the line ds round the point (x, y, z), the resultant of the forces produces no effect in the plane touching the surface, and consequently its whole action is perpendicular to that plane. The nature of the case requires further, that the same resultant be directed towards the surface of the fluid.

What has been deduced from the algebraic expressions is evident in another view. For, could we suppose that the resultant of the forces is not at every point perpendicular to the surface at liberty, it might be resolved into two partial forces, one acting in the tangent plane, and the other perpendicular to that plane; and as the first force is opposed by no obstacle, it would cause the particles to move, which is contrary to the equilibrium.

If we suppose that p is constant in the general formula of the hydrostatic pressure, we shall have an equation,

$$\varphi = C - p,$$

which is exactly similar to that of the surface at liberty, and which will determine an interior surface at every point of which there is the same intensity of

pressure. By differentiating the equation of the interior surface, we obtain

$$X\,dx + Y\,dy + Z\,dz = 0;$$

from which we deduce, by the like reasoning as before, that such surfaces are perpendicular to the resultant of the accelerating forces urging the particles contained in them. The interior surfaces in question were named level surfaces by CLAIRAUT; and they are distinguished by the two properties of being equally pressed at all their points, and of cutting the resultant of the forces at right angles. They spread through the mass, and ultimately coincide with those parts of the outer surface which are at liberty. It may be observed, that what essentially constitutes a level surface is its equation, which must differ from the equation of the outer surface at liberty in no respect, except that the constant $C - p$ takes the place of the constant C; for we shall afterwards find that, in some cases of the equilibrium of a fluid, the two properties of being equally pressed, and of cutting the resultant of the forces at right angles, belong to more sets of interior surfaces than one.

4. In what goes before, we have supposed that the density is constant, but it is easy to extend the investigation to heterogeneous fluids. Let ϱ be put for the function of the co-ordinates which expresses the variable density; then admitting that ϱ has the same value at every point of the small elementary cylinder or prism, we shall have

$$dm = \varrho\,\omega\,\delta s;$$

but, f being the whole accelerating force, urging every particle of dm in the direction of δs, we have

$$f = X\frac{\delta x}{\delta s} + Y\frac{\delta y}{\delta s} + Z\frac{\delta z}{\delta s};$$

wherefore,

$$f\,dm = \varrho\,\omega\,(X\,\delta x + Y\,\delta y + Z\,\delta z).$$

The equation expressing that the action of the accelerating forces is equal and opposite to the variation of pressure, is the same as before, viz.

$$\omega \times \delta p + f\,dm = 0;$$

and by substituting the value of $f\,dm$, we deduce

$$\delta p + \varrho\,(X\,\delta x + Y\,\delta y + Z\,\delta z) = 0. \tag{3}$$

This equation must hold at every point of the mass of fluid without any relation being supposed between the variations, wherefore p must be a function of three independent variables; and in consequence the foregoing equation implies the three separate equations following, viz.

$$\frac{dp}{dx} = -\varrho X, \quad \frac{dp}{dy} = -\varrho Y, \quad \frac{dp}{dz} = -\varrho Z.$$

It now appears that the conditions of integrability must be fulfilled, viz.

$$\frac{d.\varrho X}{dy} = \frac{d.\varrho Y}{dx}, \quad \frac{d.\varrho X}{dz} = \frac{d.\varrho Z}{dx}, \quad \frac{d.\varrho Y}{dz} = \frac{d.\varrho Z}{dy}:$$

and unless the forces possess the properties expressed by these equations, the equilibrium will be impossible.

Without pursuing the investigation in all its generality, we shall confine our attention to the case in which

$$X\,dx + Y\,dy + Z\,dz,$$

is an exact differential; a supposition that comprehends all the applications of the theory. If we represent the integral of the differential by φ, so that

$$d\varphi = X\,dx + Y\,dy + Z\,dz;$$

and convert the variations of equation (3) into differentials, we shall obtain

$$dp + \varrho\,d\varphi = 0;$$

and hence

$$p = C - \int \varrho\,d\varphi. \tag{4}$$

From this we deduce the equation of those parts of the outer surface which are at liberty, by making $p = 0$; and that of a level surface, by assigning to p some constant value. And if we differentiate the same equation (4) on the supposition that p is invariable, we shall get

$$\varrho\,d\varphi = \varrho\,(X\,dx + Y\,dy + Z\,dz) = 0,$$

which differential equation is common to the outer surface at liberty, and to all the interior level surfaces; and from which we deduce by the like reasoning as before, that all such surfaces are perpendicular to the resultant of the accelerating forces urging the particles contained in them.

The quantity under the sign of integration in the formula,

$$p = C - \int \varrho\,d\varphi,$$

must be an exact differential, for p must be a function of the co-ordinates; which condition will not be fulfilled unless ϱ be a function of φ. Thus both the pressure p and the density ϱ are functions of the same quantity φ, and they are both invariable where φ is constant. The density is therefore the same at all the points of any level surface. If we conceive a heterogeneous fluid in equilibrium to be divided into thin strata by level surfaces infinitely near one another, the density will be the same throughout every stratum, but it will vary from one stratum to another.

5. We have now placed before the reader the general points of the theory of the equilibrium of fluids. What has been said comprehends all that can be determined when a fluid is conceived to extend indefinitely; but in applying the theory to limited masses, it is necessary besides, that the pressures propagated through the interior parts either be supported or mutually balance one another.

In treating of the equilibrium of fluids, another mode of investigation is sometimes employed, which it would be improper to pass by without notice, as it is useful on many occasions to fix the imagination, although it leads to no new results. We allude to the narrow canals supposed to traverse the mass in various ways, of which so much use has been made by CLAIRAUT and other authors.

Let two points $(x^{\circ}, y^{\circ}, z^{\circ})$ and (x', y', z') be assumed in the interior of a mass of fluid in equilibrium, and conceive an infinitely narrow canal of any figure to pass between them; we may suppose that the whole fluid, except the portion within the canal, becomes solid without any change taking place in the position of the particles, or in their mutual action upon one another; for, as this supposition makes no alteration of the forces urging the particles contained within the canal, these particles will remain at rest after the solidification as they were at first. Suppose that the canal is divided into infinitely small parts by sections perpendicular to its sides; at any point (x, y, z) let ω be the section; δs the infinitely small part of the length of the canal; dm the quantity of matter in the length δs, that is, the product of the volume and the density, or $\varrho \times \omega \times \delta s$; and f the sum of all the partial forces that urge the particles of dm in the direction of the canal; then, the motive force of dm, or its effort to move, will be equal to $f\,dm$. Further, p being the hydrostatic

pressure at the point (x, y, z), the like pressure at the distance of δs will be $p + \delta p$; therefore the opposing pressures which act upon the two ends of the part of the canal in the length δs, will be $p \times \omega$ and $(p + \delta p) \times \omega$; and $\delta p \times \omega$ will be the effective pressure which pushes $d m$ towards the point (x, y, z). Because every part of the canal is supposed at rest, the tendencies of $d m$ to move in opposite directions must be equal, and we shall have this equation,

$$\delta p \times \omega + f\, d m = 0;$$

consequently,

$$\delta p + \frac{f\, d m}{\omega} = 0;$$

and by taking the sum of the similar quantities in all the parts of the canal, we obtain

$$\int \delta p + \int \frac{f\, d m}{\omega} = 0.$$

But p being a function of three independent variables, the sum of its variations, supposing the flowing quantities to follow any arbitrary law of increase or decrease, is equal to the difference of p' and p°, the final and initial values of the function; wherefore we have

$$p' - p^{\circ} + \int \frac{f\, d m}{\omega} = 0.$$

Now $f\, d m$, that is the quantity of matter multiplied by the accelerating force, is the impulse or pressure in the direction of the canal caused by all the forces urging $d m$; and as this pressure is exerted on the surface ω, $\frac{f\, d m}{\omega}$ is the same pressure reduced to the unit of surface. Therefore, whatever be the figure of the canal, it follows from the foregoing investigation, that the difference of the pressures at its two extremities is equal to the sum of the impulses of all the contained molecules of fluid, every impulse being reduced to the direction of the canal and to the unit of surface.

If the extremities of the canal be both in the parts of the outer surface which are at liberty, the pressures p' and p° will be both evanescent, and there will be no effort of the fluid either way, and no tendency to run out at one end. Further, if a canal be continued through the fluid till it return into itself, the

initial and final pressures being the same, the impulses of the molecules in the whole circuit will balance one another. But in this case, the reasoning we have employed will not be exact, unless p, the algebraic expression of the pressure, be such a function as admits of only one value for any three given co-ordinates; a restriction however, which, in every point of view, seems indispensable.

6. The whole theory, it will readily appear from the foregoing investigations, is built on the assumption, That the hydrostatic pressure at every point of the fluid is the same function of the co-ordinates of the point. The accelerating forces are represented by the partial differential coefficients of the pressure; and therefore they are likewise the same functions of the co-ordinates of their point of action in every part of the mass. The whole reasoning rests on these fundamental points; and if the state of a fluid were such that they are not verified, the equations for determining the required figure could not be formed, and the equilibrium would be impossible. As the hydrostatic pressure is known only by means of the given accelerating forces, it seems most suitable to employ the properties of the latter in laying down what is required for the equilibrium of a mass of fluid. It is necessary, and it is sufficient for the equilibrium of a homogeneous fluid, first, that the accelerating forces acting in the directions of the co-ordinates be, in every part of the mass, the same functions of the co-ordinates; and, secondly, that these functions possess the conditions of integrability. When these two conditions are both fulfilled, the determination of the figure of equilibrium is reduced to a question purely mathematical. For we can form the equation (1) which makes the accelerating forces balance the variation of pressure; and, by integrating this equation, we obtain the hydrostatic pressure, from which is deduced the equation of all those points at which there is no pressure, or in other words, the equation of all those parts of the outer surface which are at liberty. Nothing more is required for securing the permanence of the figure of the fluid, except that the pressures propagated through the mass be either supported or mutually balance one another.

The conditions for the equilibrium of a homogeneous fluid, as they are here laid down, do not enable us in all cases to form immediately the equation of the figure of equilibrium. If the particles attract or repel one another, the accelerating forces will, for the most part, vary as the fluid changes its form,

and they may not be at every point the same functions of the co-ordinates in all the figures, of which it is susceptible; but, notwithstanding the equilibrium may still be possible, because this indispensable condition may be fulfilled when figures of a certain class are induced on the mass. In such cases, the determination of the equilibrium necessarily requires two distinct researches; of which one is to find out what are the particular figures into which the mass must be moulded, so as to make the accelerating forces at every point the same functions of the co-ordinates. After these figures have been found, we can apply to them the equations expressing the conditions of equilibrium, and accomplish the mathematical solution of the problem. But if it shall appear that no figure whatever capable of fulfilling both the conditions laid down above can be induced on the fluid, the equilibrium will be absolutely impossible.

In the usual exposition of this theory, the equilibrium is made to depend on conditions that do not exactly coincide with those at which we have arrived. According to Clairaut and all other authors who have written on this subject, it is necessary, and it is sufficient, for the equilibrium of a homogeneous fluid, first, that the expressions of the accelerating forces possess the criterion of integrability; secondly, that the resultant of the forces in action at all the parts of the outer surface which are at liberty, be directed perpendicularly towards these surfaces. We may throw out of view what regards the criterion of integrability, about which there is no difference of opinion, and which in reality is always fulfilled by the forces that occur in physical researches. The perpendicularity of the forces to the outer surface is a property of the differential equation of that surface, and will necessarily take place whenever it is possible to form that equation. Nothing more is required for forming the equation mentioned, than that the accelerating forces at every point of it be expressed by the same functions of the co-ordinates of the point.* It follows

* The forces are perpendicular to every surface in which the pressure is constant. The outer surfaces are those at every point of which there is no pressure. In all the questions that have occurred, the forces at the outer surface of the fluid are the same functions of the co-ordinates of the point, whatever geometrical figure the fluid is supposed to assume; and on this account the equation of the outer surface can be formed without reference to any particular class of figures. But this is not sufficient; for, according to the fundamental assumption laid down by Clairaut himself, the theory of equilibrium cannot be applied, unless the forces be the same functions of the co-ordinates of their point of action in every part of the mass.

therefore, that the difference between the conditions of equilibrium hitherto universally adopted, and those laid down above, amounts to this: according to the former it is required that the expressions of the accelerating forces be the same functions of the co-ordinates at every point of the outer surface, this being all that is necessary for forming the differential equation of that surface; according to the latter, the forces will not balance the pressure, and the laws of equilibrium will not be fulfilled unless the forces be the same functions of the co-ordinates at every point whether situated in the outer surface, or in the interior part of the mass.

If a homogeneous fluid, of which the particles are urged by accelerating forces be in equilibrium, all that is required by Clairaut's theory will undoubtedly be fulfilled; but the converse of this cannot be affirmed. It is no where proved generally by unexceptionable arguments, and indeed no proof can possibly be given, that the forces in the interior parts of the fluid will balance the pressure, merely because the resultant of the forces in action at the outer surface is perpendicular to that surface. All the attempts that have been made to demonstrate this point, tacitly assume that the expression of the forces is the same at the surface and in all the interior parts; which is not universally true.

In a very extensive class of problems the difference between the two ways of laying down the conditions of equilibrium disappears. This will happen when the accelerating forces are independent of the figure of the fluid, as will be the case if the particles exert no action on one another by attraction or repulsion. In such problems the forces impressed upon every particle, whatever be its situation, and whatever be the figure of the fluid, are by the hypothesis, the same given functions of the co-ordinates. The figure of equilibrium will be the same whether, following Clairaut, we obtain the equation of the outer surface by means of the forces in action at that surface, or, making use of the property that the pressure vanishes at all the points where the fluid is at liberty, we deduce the same equation from the pressure that prevails generally throughout the mass.

But Clairaut's theory cannot be extended to the solution of other problems than those of which we have been speaking. In no other cases is it evident without inquiry that the proposed accelerating forces urging a particle, are, in

every part of the mass, the same functions of the coordinates of the particle; and unless this be verified, the theory of equilibrium cannot be applied. In a homogeneous planet in a fluid state, there are forces which prevail in the interior parts and vanish at the surface; and, as CLAIRAUT's theory notices no forces except those in action at the surface, it leaves out some of the causes tending to change the figure of the fluid, and therefore it cannot lead to an exact determination of the equilibrium.

II. *Application of the foregoing Theory to the Question of the Figure of the Planets.*

7. Having now explained the general theory of the equilibrium of fluids at sufficient length, I proceed to apply it to the question of the figure of the planets, in which it is required to determine the equilibrium of a fluid entirely at liberty, and unconfined by any obstacle or support. The problem is one of considerable difficulty. It is necessary to distribute the investigation under distinct heads. It would otherwise be impossible to preserve perspicuity and precision of ideas in an inquiry essentially different in different hypotheses. The equilibrium of a homogeneous fluid must occupy our attention before that of one having its density variable. For although it may at first appear that the latter problem is the more general, and includes the former, yet it will be found that the equilibrium of a fluid of variable density, depends upon that of a homogeneous fluid, and is deducible from it. And even with regard to homogeneous fluids, distinctions must be made, because what is required for the equilibrium varies with the nature of the accelerating forces. In this respect we distinguish these two general cases, of which we shall treat in two separate problems; First, when the accelerating forces depend only on the coordinates of their point of action, and are explicitly known when the coordinates are given; Secondly, when the accelerating forces depend not only upon the coordinates of the particle on which they act, but likewise upon the figure of the whole mass of fluid; as happens for the most part when the particles attract or repel one another.

Problem 1st.—To determine the equilibrium of a homogeneous mass of fluid which is entirely at liberty, when the accelerating forces are known functions of the coordinates of their point of action.

The equilibrium of a mass of fluid which is entirely at liberty, can depend only upon the action of such forces as tend to change the relative position of the particles with respect to one another. It is not affected by any motion common to all the particles, nor by any force which acts upon them all with the same intensity in the same direction; the effect of such motion, or of such force, being to displace the centre of gravity of the whole mass without altering the relative situation of the particles. In estimating the accelerating forces upon which the figure of equilibrium will depend, we must therefore begin with reducing the centre of gravity, if it be in motion or urged by any force, to a state of relative rest; which is accomplished by applying to every particle a force that would cause it to move with the same velocity as the centre of gravity, but in a contrary direction. In the investigation of this problem we may therefore suppose that the centre of gravity is at rest and undisturbed by the action of any accelerating force.

Suppose now that a mass of homogeneous fluid entirely at liberty, is in equilibrium, and conceive three planes intersecting at right angles in the centre of gravity of the mass, to which planes the particles of the fluid are to be referred by rectangular coordinates. Let x, y, z, represent the coordinates of a particle, and having resolved the accelerating forces acting upon it into other forces that have their directions parallel to the coordinates, put X, Y, Z, for the sums of the resolved parts respectively parallel to x, y, z, and tending to shorten these lines. According to the hypothesis of this problem, the forces X, Y, Z, depend only upon the coordinates of their point of action; and they are at every point the same functions of those coordinates. The equilibrium will therefore be impossible unless

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz$$

be an exact differential, this being necessary in order that the hydrostatic pressure be a function of three independent variables as the fundamental assumption of the theory demands. Let φ denote the integral, and p the

hydrostatic pressure at the point $(x, y, z,)$: the equations that determine the equilibrium will be these two *,

$$\left.\begin{array}{l} \varphi = \int (X\,dx + Y\,dy + Z\,dz), \\ p = C - \varphi. \end{array}\right\}$$

If we make $p = 0$, we shall obtain the equation of the outer surface of the fluid, viz.

$$\varphi = C.$$

The differential equation,

$$\frac{d\varphi}{dx}\,dx + \frac{d\varphi}{dy}\,dy + \frac{d\varphi}{dz}\,dz = 0,$$

or which is the same,

$$X\,dx + Y\,dy + Z\,dz = 0,$$

is common to the outer surface and to all the interior level surfaces at every point of which there is the same intensity of pressure; and it shows that the resultant of the accelerating forces is perpendicular to all such surfaces †.

The figure of the fluid being determined, it remains to inquire whether the equilibrium is secured. By varying the coordinates in the formula for p, we obtain

$$\delta p + \frac{d\varphi}{dx}\delta x + \frac{d\varphi}{dy}\delta y + \frac{d\varphi}{dz}\delta z = 0:$$

which equation proves that, if a particle be moved from its place a very little in any direction, the variation of the intensity of pressure is equal and opposite to the action of the accelerating forces. A particle has therefore no tendency to move from inequality of pressure. But we must not from this hastily conclude that there is no cause tending to change the figure of the fluid. For, as in the simple case of a fluid contained in a vessel, the equilibrium requires not only that the accelerating forces balance the inequality of pressure, but likewise that the total pressures tending outward at the boundaries of the mass, be supported by the sides of the vessel; so in the problem under consideration, there being no external support, the figure of the fluid will not be permanent

* Equation (2) § 2. † Equation (2) § 3.

unless the pressures propagated inward, which increase as any point sinks deeper below the surface, mutually compensate and destroy one another. Some further discussion is therefore necessary in order to prove that the equilibrium is completely established.

The function φ, in which we may suppose there is no constant quantity, can contain no term having the coordinates for divisors; for, were this the case, the pressure would be infinite at all those points where such coordinates are equal to zero. Let the terms of φ be arranged in homogeneous expressions of one, two, three, &c. dimensions; then

$$\begin{aligned}\varphi = {} & (A_1 x + A_2 y + A_3 z) \\ & + (B_1 x^2 + B_2 y^2 + B_3 z^2 + B_4 x y + B_5 x z + B_6 y z) \\ & + (D_1 x^3 + D_2 y^3 + D_3 z^3 + D_4 x^2 y + \&c.) \\ & + \&c.\end{aligned}$$

Differentiate this expression, and after the operations put $x = 0$, $y = 0$, $z = 0$: then

$$\frac{d\varphi}{dx} = A_1, \quad \frac{d\varphi}{dy} = A_2, \quad \frac{d\varphi}{dz} = A_3.$$

But the differentials of φ are no other than the expressions of the accelerating forces acting on a particle; consequently A_1, A_2, A_3 are the forces in action at the origin of the coordinates, that is, at the centre of gravity of the mass. Wherefore, according to what was observed, we shall have

$$A_1 = 0, \quad A_2 = 0, \quad A_3 = 0,$$

$$\begin{aligned}\varphi = {} & (B_1 x^2 + B_2 y^2 + B_3 z^2 + B_4 x y + B_5 x z + B_6 y z) \\ & + (D_1 x^3 + D_2 y^3 + D_3 z^3 + D_4 x^2 y + \&c.) \\ & + \&c.\end{aligned}$$

That the expression of φ must be of this form is required by the nature of the problem: for φ must be always positive, and it must increase continually from the centre of gravity to the surface of the fluid.

Let us now put

$$\begin{aligned} x &= r \cos\theta = r\xi, \\ y &= r \sin\theta \cos\psi = r\eta, \\ z &= r \sin\theta \sin\psi = r\zeta, \end{aligned}$$

then r will be the line drawn from the centre to the point (x, y, z); and the arcs θ and ψ determine the direction of r, θ being the angle between r and the axis of the coordinates parallel to x, and ψ the angle which the plane containing r and the same axis makes with the plane of x, y. By substituting, we get

$$\begin{aligned}\phi = {} & r^2 (B_1 \xi^2 + B_2 \eta^2 + B_3 \zeta^2 + B_4 \xi \eta + B_5 \xi \zeta + B_6 \eta \zeta) \\ & + r^3 (D_1 \xi^3 + D_2 \eta^3 + D_3 \zeta^3 + D \xi^2 \eta + \&c.) \\ & + \&c.\end{aligned}$$

The symbols ξ, η, ζ, represent three rectangular coordinates of a point in the surface of a sphere having unit for its radius; and, in order to simplify, I shall write Q_2, Q_3, and generally Q_n, for homogeneous functions of ξ, η, ζ, of two, three, and n dimensions: then,

$$\phi = r^2 Q_2 + r^3 Q_3 + r^4 Q_n + \&c.$$

For the sake of distinction, let R represent a line drawn from the centre of gravity to the surface of the fluid; and r a line drawn from the same centre to any interior point at which the pressure is p, the directions in which R and r are drawn being determined by the arcs θ and ψ: the equation of the fluid's surface, and the expression of p, will be as follows,

$$\begin{aligned} C &= R^2 Q_2 + R^3 Q_3 + R^4 Q_n + \&c. \\ p &= C - (r^2 Q_2 + r^3 Q_3 + r^4 Q_n + \&c.)\end{aligned}$$

By means of these equations a radius, R or r, will be known when the arcs θ and ψ which determine its direction are assumed; and in this manner we may find all the points of the outer surface, and of any interior level surface in which p has any assigned value less than C. All these surfaces will return into themselves and inclose a space: because in whatever direction we proceed from the centre of gravity to the surface, the function ϕ passes through every gradation of magnitude between zero and the maximum.

It is now easy to complete the demonstration of the equilibrium. A stratum of the fluid between the outer surface and any interior level surface will evidently be in equilibrium, if we suppose that the level surface maintains its figure, or rather, that there are no forces urging the particles contained within that surface: for, the upper part of the stratum cuts the resultant of the forces

at right angles, and the fluid presses perpendicularly and with the same intensity at every point of the lower surface which supports the stratum. What is here affirmed is true, however near the level surface be to the centre of gravity; and as the accelerating forces urging the particles within the surface decrease without limit in approaching that centre, they may finally be regarded as evanescent when the internal body of fluid is no more than a drop occupying the centre of gravity. Wherefore, by taking the radius of the level surface small enough, the inclosed fluid may be considered free from any accelerating forces, and subject only to the external pressures; and, these being perpendicular to the surface, and acting with the same intensity, the whole mass of fluid will be in equilibrium by the known laws of hydrostatics.

It may be proper to add that the mass of fluid has no tendency to turn upon an axis. For no motion of this kind can be produced by the pressures propagated inward from the surface, the directions of which pass through the centre of gravity. Neither can the accelerating forces urging the particles, cause any such motion, these being wholly employed in counteracting the inequality of pressure.

For the sake of illustrating the problem we have solved, we shall add one example, which is besides intimately connected with the principal subject of our research.

Example.—To determine the figure of equilibrium of a homogeneous mass of fluid entirely at liberty, the particles being supposed to attract one another with a force directly proportional to the distance at the same time that they are urged by a centrifugal force caused by rotation about an axis.

At first view the proposed problem may seem one in which the accelerating forces depend upon the figure of the fluid, since it is supposed that every particle is attracted by every other. But, in the particular law of attraction assumed, the force which urges any particle is directed to the centre of gravity of the whole mass of matter, and is proportional to the distance from that point*. The hypothesis of the problem is therefore equivalent to the supposition that the particles of the fluid are attracted to a fixt centre with a force proportional to the distance; so that the accelerating forces are independent of the figure of the fluid.

* Prin. Math. Lib. i. Prop. 88.

As the centre of gravity of a mass of fluid in equilibrium must be free from the action of any force, except what is common to all the particles; and as the attractions of the particles balance one another at that point; the centrifugal force must likewise be evanescent at the same point, and consequently the axis of rotation must pass through it. Let three planes intersecting at right angles, one being perpendicular to the axis of rotation, pass through the centre of gravity; and assuming any particle of the fluid, let r denote its distance from the same centre, and x, y, z its coordinates, z being parallel to the axis of rotation: further, let g represent the attractive force of the whole mass of fluid at the distance equal to unit from the centre of gravity; and f the centrifugal force (that is, its proportion to g) at the distance equal to unit from the axis of rotation: then $g\,r$ will be the central attraction urging the particle, and $g\,x$, $g\,y$, $g\,z$, will be the resolved parts of the same force in the directions of the coordinates: also, $\sqrt{x^2+y^2}$ will be the distance of the particle from the axis of rotation; $-f\sqrt{x^2+y^2}$, the whole centrifugal force estimated as tending to shorten the coordinates; and $-f x$, $-f y$, the resolved parts of the same force, parallel to x and y: collecting, now, the partial forces which urge the particle in the respective directions of the coordinates, we shall find,

$$\mathrm{X} = (g-f)\,x, \qquad \mathrm{Y} = (g-f)\,y, \qquad \mathrm{Z} = g\,z.$$

The equations of equilibrium will, therefore, be

$$\phi = \int (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz) = \tfrac{1}{2}\{(g-f)(x^2+y^2) + g\,z^2\},$$
$$p = \mathrm{C} - \tfrac{1}{2}\{(g-f)(x^2+y^2) + g\,z^2\}$$

The equation of the surface of the fluid will be found by making $p = 0$, viz.

$$\mathrm{C} = \tfrac{1}{2}(g-f)(x^2+y^2) + \tfrac{1}{2}g\,z^2.$$

And, if we put $e^2 = \frac{f}{g}$, the same equation may be thus written,

$$a^2 = x^2 + y^2 + \frac{z^2}{1-e^2},$$

which belongs to an elliptical spheroid of revolution having the equatorial semidiameter equal to a, and the polar semi-axis to $a\sqrt{1-e^2}$.

8. The order of discussion that has been laid down now brings us to the

more difficult part of this research, when the accelerating forces urging the particles of the fluid, depend upon the very figure of equilibrium which is to be investigated. This must happen in fluids consisting of particles that mutually attract one another, if the attractive force acting upon a particle vary with the figure of the attracting matter. In this division of our subject, the law of attraction that prevails in nature being in reality the only one which it is of much importance to consider, will chiefly engage attention.

Problem 2nd.—To determine the equilibrium of a homogeneous fluid entirely at liberty, the particles attracting one another with a force inversely proportional to the square of the distance, at the same time that they are urged by a centrifugal force caused by rotation about an axis.

The fluid being supposed in equilibrium, the axis of rotation must pass through the centre of gravity of the mass. For, abstracting from any motion or force common to all the particles, that centre may be considered at rest and free from the action of any accelerating force; and, as the attractive forces balance one another at that point, the centrifugal force must likewise vanish at the same point.

Conceive three planes intersecting at right angles in the centre of gravity of the mass, one of them being perpendicular to the axis of rotation: let x, y, z represent the coordinates of a particle in the surface of the fluid, x being parallel to the same axis; and put V for the sum of the quotients of all the molecules of the mass divided by their respective distances from the particle: then the attractive forces urging the particle inward in the directions of x, y, z, will be respectively equal to

$$-\frac{dV}{dx}, \quad -\frac{dV}{dy}, \quad -\frac{dV}{dz}.$$

Further, if f denote the centrifugal force at the distance unit from the axis of rotation, the action of the same force at the distance $\sqrt{y^2+z^2}$ from the same axis will be $f\sqrt{y^2+z^2}$; and the resolved parts of this force urging the particle to move in the prolongations of y and z, will be fy and fz. Wherefore the total forces parallel to x, y, z, and tending to shorten these lines, are respectively,

$$-\frac{dV}{dx}, \quad -\left(\frac{dV}{dy}+fy\right), \quad -\left(\frac{dV}{dz}+fz\right):$$

and the condition that the resultant of these forces is perpendicular to the surface of the fluid is expressed by this differential equation,

$$\frac{dV}{dx}dx+\frac{dV}{dy}dy+\frac{dV}{dz}dz+f(y\,dy+z\,dz)=0\,;$$

and the integral, viz.

$$C=V+\frac{f}{2}(y^2+z^2),$$

is the equation of the surface of the fluid in equilibrium. This is incontestably the true equation of the surface in equilibrium, since all the forces in action at that surface have been taken into account.

Using x, y, z to represent generally the co-ordinates of any particle of the mass, and the symbol V, to denote the function of x, y, z, which is equal to the sum of the quotients of all the molecules of the mass of fluid divided by their respective distances from the particle, it will be convenient to have some means of pointing out whether V belongs to a point in the surface, or to one differently situated. For this purpose we shall put $r=\sqrt{x^2+y^2+z^2}$ for the distance from the centre of gravity, and shall write V (r) for the value of V relatively to a point within the mass; and we shall suppose that r becomes R at the upper surface, so that V (R) will denote the value of V for a point in that surface. According to this notation, the foregoing equation of the surface of the fluid in equilibrium, will be thus written,

$$C=V(R)+\frac{f}{2}(y^2+z^2). \tag{1}$$

The attraction of the whole mass and the centrifugal force, which are the only forces that urge a particle in the upper surface, likewise act upon every particle in the interior parts of the fluid. It will contribute to perspicuity if to these forces we give the name of the *principal forces*, in order to distinguish them from any other forces which an attentive examination may enable us to detect. Assuming any molecule in the interior parts, r being its distance from the centre of gravity, and x, y, z its coordinates, we have only to proceed as before, writing V (r) for V, in order to find the resolved parts of the principal

forces which urge the molecule inward in the respective directions of x, y, z, viz.

$$-\frac{d\,\mathrm{V}(r)}{d\,x}, \qquad -\left(\frac{d\,\mathrm{V}(r)}{d\,y}+fy\right), \qquad -\left(\frac{d\,\mathrm{V}(r)}{d\,z}+fz\right):$$

and if these forces be multiplied, each by the variation of its direction, the sum of the products will be the variation of the intensity of pressure, which is equal and opposite to their action, according to equation (1) of the general theory; thus, we have,

$$\delta p - \frac{d\,.\,\mathrm{V}(r)}{d\,x}\delta x - \frac{d\,.\,\mathrm{V}(r)}{d\,y}\delta y - \frac{d\,.\,\mathrm{V}(r)}{d\,z}\delta z - f(y\,\delta y + z\,\delta z) = 0; \qquad (2)$$

and, as this equation is true at every point of the mass, we further obtain

$$p = \mathrm{V}(r) + \frac{f}{2}(y^2+z^2) - \mathrm{C}, \qquad (3)$$

the constant being the same as in the equation (1) of the upper surface, because the two equations must coincide when the interior molecule ascends to the surface. It must be observed that p represents the intensity of pressure caused by the principal forces alone, and not the whole pressure upon the molecule, if besides these forces there exist other causes of pressure in the interior parts.

From the nature of the function V or V (r), it has its maximum at the centre of gravity of the mass, or when $r=0$; for at that point we have the equations

$$\frac{d\,.\,\mathrm{V}(r)}{d\,x}=0, \qquad \frac{d\,.\,\mathrm{V}(r)}{d\,y}=0, \qquad \frac{d\,.\,\mathrm{V}(r)}{d\,z}=0,$$

because the attractive forces balance one another. While r, without any change in its direction, increases to be equal to R, V (r) continually decreases. In whatever direction the radius R be drawn to the surface, there is always a point in it, the coordinates of which will satisfy equation (3), supposing that p has any assigned value less than the maximum which takes place at the centre of gravity. All the points in which p has the same given value will form an interior surface, returning into itself and pressed with equal intensity by the action of the principal forces upon the exterior fluid. Such interior surfaces are likewise perpendicular to the resultant of the principal forces

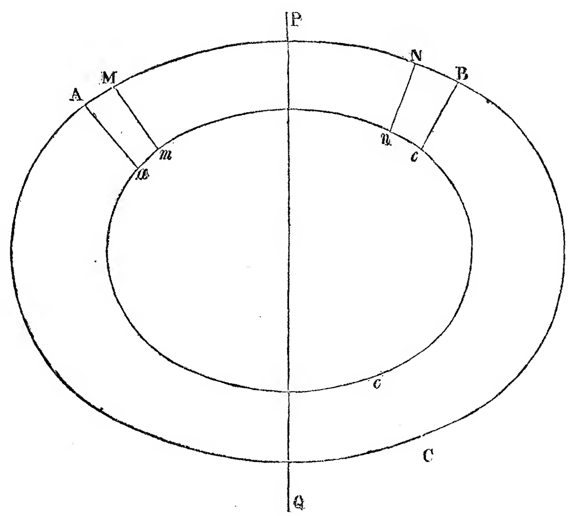
urging the particles contained in them, as will readily be proved by differentiating equation (3), making p constant.

In order to place what has been said in the clearest light, let A B C represent the mass of fluid, the surface being determined by the equation,

$$C = V(R) + \frac{f}{2}(y^2 + z^2);$$

and suppose that $a\,b\,c$ is an interior surface, obtained by making p constant in the equation,

$$p = V(r) + \frac{f}{2}(y^2 + z^2) - C:$$



then, if the narrow canal A $a\,m$ M stand upon the molecule $a\,m$ of the interior surface, and extend to the upper surface of the fluid, the intensity of pressure upon $a\,m$, or the given quantity p, will be equal to the sum of all the impulses caused by the action of the principal forces upon the molecules contained in the canal, every impulse being reduced to the direction of the canal and to the unit of surface. The same thing is true of any other molecule in the same surface upon which there stands a similar canal B $b\,n$ N.

If we attend to the conditions of equilibrium required by the general theory, it will readily appear that the equilibrium of the mass A B C will be impossible, if at any point, as $a\,m$, of the interior surface $a\,b\,c$, any other pressure exist besides that represented by p, or any other forces be in action besides those expressed by the coefficients of the variations in equation (2). For, at the upper surface, there are no forces in action but the principal forces, and the equilibrium will be impossible if other forces prevail in the interior parts besides the principal forces. On the other hand, the matter contained in the stratum between the two surfaces will attract every particle, as $a\,m$, situated in the interior surface. The attraction of the stratum is an indelible force not

to be destroyed, which will produce its full effect according to the figure and quantity of the attracting matter and the situation of the attracted point. The equilibrium will therefore be absolutely impossible, unless such a figure can be induced on the mass of fluid as will set free every particle in the surface $a\,b\,c$ *from the attraction* of the stratum. If such a figure can be found, every molecule of the mass will be urged by the principal forces only; because a surface such as $a\,b\,c$, at every point of which these forces alone will be in action, may be described through any interior molecule $a\,m$ arbitrarily assumed. We must therefore turn our attention to investigate such figures, if there be any, as will make the irregular attraction in the interior parts disappear, so as to leave the principal forces alone in action; for, unless this can be effected, the fluid cannot maintain a permanent form.

According to the notation we have used, if r denote the distance of $a\,m$ from G, $V(r)$ will represent the sum of the quotients of all the molecules of the whole mass divided by their respective distances from $a\,m$; let $V'(r)$ denote the same thing, relatively to the interior mass $a\,b\,c$, that $V(r)$ does, relatively to the whole mass A B C; then $V(r) - V'(r)$ will denote the sum of the quotients of all the molecules of the stratum divided by their respective distances from $a\,m$. Take a point $(x + dx,\ y + dy,\ z + dz)$ in the surface $a\,b\,c$ infinitely near $a\,m$; and, differentiating in the surface, the expressions,

$$\frac{d.\big(V(r) - V'(r)\big)}{dx}, \qquad \frac{d.\big(V(r) - V'(r)\big)}{dy}, \qquad \frac{d.\big(V(r) - V'(r)\big)}{dz},$$

will be equal to the attractive forces of the stratum upon the particles of $a\,m$, in the respective directions of x, y, z: but, as we have shown, the equilibrium indispensably requires that these attractions be evanescent, so that we have these equations,

$$\frac{d.\big(V(r) - V'(r)\big)}{dx} = 0, \quad \frac{d.\big(V(r) - V'(r)\big)}{dy} = 0, \quad \frac{d.\big(V(r) - V'(r)\big)}{dz}^{*} = 0,$$

* The perpendicularity to the surface $a\,b\,c$, of the attraction of the stratum upon $a\,m$, is expressed by this equation,

$$\frac{d.\big(V(r) - V'(r)\big)}{dx}\,dx + \frac{d.\big(V(r) - V'(r)\big)}{dy}\,dy + \frac{d.\big(V(r) - V'(r)\big)}{dz}\,dz = 0;$$

and it is a consequence of the differential equations in the text. The neglect of this consideration, and the assumption that the level surfaces depend solely upon the outer surface in every case, is the great blemish of CLAIRAUT's theory.

which are no other than the partial differentials of the equation,

$$V(r) - V'(r) = \text{constant.} \tag{4}$$

This equation must hold at every point of every interior surface, such as $a\,b\,c$; and, as its differentials are separately equal to zero, it must not contain the coordinates of the surface. If such a figure can be induced on the mass of fluid as will possess the property expressed by equation (4), every particle of the mass will be urged by the principal forces alone, the equilibrium will be possible, and it will be determined in the very same manner as in the first problem.

We have now obtained a mathematical property that distinguishes the figures with which the equilibrium is possible from all others. We have also, in another place*, investigated the figures that alone possess this property; and it appears from what is there shown, that A B C can be no other but an ellipsoid, and that every interior surface, as $a\,b\,c$, is similar to the outer surface, and similarly posited about G.

Having demonstrated that the fluid in equilibrium must be an ellipsoid, it readily follows that the axis of rotation must be one of the three axes of the geometrical figure. For, as the axis of rotation passes through G, the centre of gravity, it is a diameter of the ellipsoid; and the centrifugal force being evanescent at the extremities of this diameter in the surface of the fluid, the only force in action at those points is the attraction of the mass of matter. But the whole force urging every particle in the outer surface of the mass in equilibrium, is perpendicular to that surface; wherefore, the attractive force of the ellipsoid is perpendicular to its surface at the extremities of the diameter about which the fluid revolves; and as there are no points on the surface of that geometrical figure at which the attraction of its mass is perpendicular to its surface, except the extremities of its three axes, it follows that with one or other of these, the axis of rotation of the fluid in equilibrium must coincide.

Let us now determine the relations between the axes of the ellipsoid and the centrifugal force. Of the three planes of the coordinates, one, which is perpendicular to the axis of rotation, is a principal section of the ellipsoid; and we may suppose that the other two coincide with the two remaining principal sections. We may therefore compute V (R) for a point in the surface; and by substituting this value in the equation,

* Phil. Trans. for 1824.

$$0 = V(R) + \frac{f}{2}(y^2 + z^2) - C,$$

and making the result coincide with the geometrical equation of the figure, we shall obtain the expressions of the axes in terms of the centrifugal force. But it will be more simple to use the differential equation,

$$-\frac{d.V(R)}{dx}dx - \left(\frac{d.V(R)}{dy} + fy\right)dy - \left(\frac{d.V(R)}{dz} + fz\right)dz = 0,$$

which expresses the perpendicularity of the forces to the outer surface. The quantities,

$$-\frac{d.V(R)}{dx}, \quad -\frac{d.V(R)}{dy}, \quad -\frac{d.V(R)}{dz},$$

are the attractive forces of the ellipsoid, urging a particle of the surface in directions parallel to the axes; and these forces, by the nature of the ellipsoid, are proportional to the coordinates of the point on which they act, and may be represented by $A'x$, $B'y$, $C'z$, the coefficients A', B', C' being known quantities depending upon the ratios of the axes of the ellipsoid; wherefore, these values being substituted in the differential equation, we shall have,

$$A'x\,dx + (B' - f)\,y\,dy + (C' - f)\,z\,dz = 0;$$

and by integrating,

$$x^2 + \frac{B' - f}{A'}y^2 + \frac{C' - f}{A'}z^2 = \text{constant}.$$

Now, if h, h', h'' represent the axes of the ellipsoid, h being that about which the fluid revolves, the equation of the surface of the figure will be,

$$x^2 + \frac{h^2}{h'^2}y^2 + \frac{h^2}{h''^2}z^2 = h^2;$$

and with this equation the foregoing one must be made to coincide. On account of the arbitrary constant, we have only to equate the coefficients of y^2 and z^2, and the resulting formulas may be thus written,

$$f = B' - \frac{h^2}{h'^2}A', \qquad f = C' - \frac{h^2}{h''^2}A'.$$

But, on examining the functions that A', B', C' stand for, it will readily appear that the expressions on the right side of the two formulas will not be positive,

and consequently they cannot be equal to f, unless $\frac{h^2}{h'^2}$ and $\frac{h^2}{h''^2}$ be both less than unit: and supposing that h is the least of the three axes, the two values of f will not be equal, unless $B' = C'$, and $h' = h''$, in which case both the formulas coincide in one, viz.

$$f = B' - \frac{h^2}{h'^2} A'.$$

In conclusion, it follows that the figure of the fluid in equilibrium is an oblate elliptical spheroid of revolution, of which the equation is

$$x^2 + \frac{h^2}{h'^2}(y^2 + z^2) h^2,$$

the mass turning about h the less axis, and the relation between the centrifugal force and $\frac{h}{h'}$ the ratio of the axes, being determined by the equation

$$f = B' - \frac{h^2}{h'^2} A'.$$

The complete solution of the problem is now brought to the discussion of this last equation; and as this is a question purely mathematical, but slightly connected with the physical conditions of the equilibrium, which we have undertaken to investigate, we shall refer to the Mécanique Céleste of LAPLACE and to the Théorie Analytique du Système du Monde of M. de PONTECOULANT, in which works this point is amply treated.

The foregoing solution, being perfectly general, proves that the equilibrium is possible only when the elliptical spheroid is oblate at the poles. When the spheroid is oblong, and the axis of rotation h greater than the other axis h', the expression that must be equal to the centrifugal force is negative; and as that force is essentially positive, the equilibrium becomes impossible.

It will not be necessary to retrace the steps of the foregoing analytical process of reasoning, in order to show synthetically that the equilibrium will be secured if the conditions deduced be fulfilled. For, as soon as such a figure is found as will make the forces that actually urge every particle of the mass the same functions of the coordinates of their point of action, this problem comes under the hypothesis of the first one, and may be demonstrated in the very same manner.

The method of solution we have here followed may be applied to all problems concerning the equilibrium of a mass of fluid, when it is possible to form the equation of the outer surface; that is, when the forces in action at all the points of the outer surface are the same functions of the coordinates of those points, whatever geometrical figure the mass may be supposed to assume. This in reality comprehends every question that has hitherto occurred; and, as the conditions which we have laid down are necessary and sufficient for the equilibrium in every hypothesis of the forces that can be imagined, we shall not enter into any further discussion of this point.

9. The preceding analysis, by which we have investigated the figure of equilibrium of a homogeneous planet is direct and unexceptionable in point of rigour. It seems hardly possible to express simply in algebraic language, all the forces that urge the interior particles of the fluid; and this makes it necessary to have recourse to peculiar modes of reasoning for determining the figure of equilibrium. The problem, being one of great importance and difficulty, which has much engaged the attention of geometers, and which requires for its solution principles different from those that have so long passed current without suspicion of their accuracy, it may not be improper to add another investigation of it by a process of reasoning very different from the foregoing.

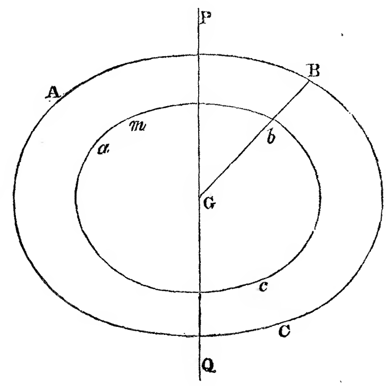
Second investigation.

We shall begin with laying down the following lemma. If a mass of homogeneous fluid, consisting of particles which attract one another inversely as the square of the distance, be in equilibrium when it revolves with a certain angular velocity about an axis; any other mass of the same fluid, the particles attracting by the same law, will be in equilibrium, if it have a similar figure, and revolve with the same rotatory motion about an axis similarly placed.

Take any two particles similarly placed in the two bodies, and having the same proportion to one another as the whole masses; it is proved in the Principia of NEWTON, and in the works of other authors, that the resultants of the attractive forces acting upon the particles, have similar directions, and are proportional to the linear dimensions of the two bodies. Further, the centrifugal forces urging the two bodies to recede from the axes of rotation, are proportional to the respective distances from the axes, that is, to the linear dimen-

sions of the two bodies. Wherefore the joint action of all the forces is to urge the two particles in similar directions with intensities proportional to the linear dimensions of the bodies. And as the same thing is true of all particles similarly situated in the two bodies, if there be an equilibrium in one case, there will be an equilibrium in the other; for the forces which urge the particles of one body are in no respect different from the forces which urge the particles of the other, except in being all increased or all diminished in the same given proportion.

This lemma being premised, let A B C represent a mass of homogeneous fluid in equilibrium, by the attraction of its particles in the inverse proportion of the square of the distance, and a centrifugal force caused by revolving about the axis P Q. The axis P Q will pass through G, the centre of gravity of the mass. For, abstracting from any motion or force common to all the particles, that centre may be considered at rest; and, as the attractive forces of the particles balance one another at that point, the centrifugal force must likewise vanish at the same point.



Let any radius G B, drawn from the centre of gravity to the surface of the fluid, be divided in a given proportion at *b*; and supposing it to turn round G so as to be directed successively to all the points in the outer surface of the fluid, the radius G *b*, being always the same part of G B, will describe an interior surface similar to the outer one, and similarly posited about G. And because the whole mass A B C is in equilibrium, it follows from the lemma that the interior mass *a b c*, which is similar to the whole mass, and revolves with it about the common axis P Q, will be separately in equilibrium, supposing the exterior stratum of matter were taken away or annihilated.

In the interior surface *a b c* assume any molecule *a m*: the forces that act upon *a m* are; first, the resultant of the centrifugal force and the attraction of the mass *a b c*; secondly, the attraction of the stratum of fluid between the two surfaces. Because the interior body of fluid *a b c* is separately in equili-

brium, the first of these forces, namely, the resultant of the centrifugal force and the attraction of the mass $a\,b\,c$, is perpendicular to the surface $a\,b\,c$, and destroyed by the resistance of the fluid within that surface; and from this it follows that the attraction of the stratum upon $a\,m$, must likewise be perpendicular to the same surface. For, if it acted obliquely to the surface $a\,b\,c$, it might be resolved into two partial forces, one perpendicular, and the other parallel, to the plane touching the surface; and as there is no obstacle to oppose the latter force, it would cause the molecule $a\,m$ to move, which is contrary to the equilibrium of the whole mass A B C. It appears therefore that two distinct and independent conditions are required for the equilibrium of the fluid mass: for all the particles situated in any interior surface $a\,b\,c$ similar to the outer surface, and similarly posited about the centre of gravity G, must be urged perpendicularly to the surface in which they are contained, not only by the resultant of the centrifugal force and the attraction of the interior mass, but likewise by the attraction of the exterior stratum of fluid.

Conceive three planes intersecting at right angles in the centre of gravity of the mass, one of them being perpendicular to the axis of rotation P Q: let x, y, z represent the coordinates of the molecule $a\,m$, and $r = \sqrt{x^2 + y^2 + z^2}$, its distance from G, x being parallel to P Q; and put V (r) for the sum of the quotients of all the molecules of the whole mass A B C, divided by their respective distances from $a\,m$: further, let V′ (r) denote the same thing relatively to the interior mass $a\,b\,c$, that V (r) does relatively to the whole mass A B C: then V (r) − V′ (r) will be the sum of the quotients of all the molecules of fluid contained in the stratum between the two surfaces, divided by the respective distances of the molecules from $a\,m$. According to the known properties of this function, the partial attractions of the stratum upon $a\,m$, in the directions of x, y, z, and tending to lengthen these lines, will be respectively equal to

$$\frac{d.\left(\mathrm{V}(r) - \mathrm{V}'(r)\right)}{d\,x}, \qquad \frac{d.\left(\mathrm{V}(r) - \mathrm{V}'(r)\right)}{d\,y}, \qquad \frac{d.\left(\mathrm{V}(r) - \mathrm{V}'(r)\right)}{d\,z}.$$

Take any point $(x + d\,x,\ y + d\,y,\ z + d\,z)$ in the surface $a\,b\,c$, at the infinitely small distance $d\,s$ from $a\,m$: then the resultant of the foregoing attracting forces in the direction of $d\,s$ will be equal to

$$\frac{d.\left(V(r)-V'(r)\right)}{dx}\cdot\frac{dx}{ds}+\frac{d.\left(V(r)-V'(r)\right)}{dy}\cdot\frac{dy}{ds}+\frac{d.\left(V(r)-V'(r)\right)}{dz}\cdot\frac{dz}{ds}:$$

and this resultant must be equal to zero in whatever direction ds is drawn, if the attraction of the stratum upon am be perpendicular to the surface abc. Wherefore we have,

$$\frac{d.\left(V(r)-V'(r)\right)}{dx}dx+\frac{d.\left(V(r)-V'(r)\right)}{dy}dy+\frac{d.\left(V(r)-V(r)\right)}{dz}dz=0: \quad (5)$$

and, by integrating,

$$V(r)-V'(r)=\text{Constant}, \quad (6)$$

which equation must be true at every point in the surface abc.

Again, the attractive forces of the interior mass urging the molecule am inwards in the direction of x, y, z, are respectively equal to

$$-\frac{d.V'(r)}{dx}, \quad -\frac{d.V'(r)}{dy}, \quad -\frac{d.V'(r)}{dz}.$$

Let f denote the centrifugal force at the distance unit from the axis of rotation; and, the distance of am from the same axis being $\sqrt{y^2+z^2}$, the centrifugal force of the particles of am will be $f\sqrt{y^2+z^2}$; and the resolved parts of this force acting in the prolongations of y and z, will be fy and fz. Wherefore the total accelerating forces urging am in the directions of x, y, z, and tending to shorten these lines, are respectively,

$$-\frac{d.V'(r)}{dx}, \quad -\left(\frac{d.V'(r)}{dy}+fy\right), \quad \left(\frac{d.V(r)}{dz}+fz\right):$$

and, the condition that the resultant of these forces is perpendicular to the surface abc, is expressed by this differential equation,

$$-\frac{d.V'(r)}{dx}dx-\left(\frac{d.V'(r)}{dy}+fy\right)dy-\left(\frac{d.V'(r)}{dz}+fz\right)dz=0 \quad (7)$$

In the equations (5) and (7) the forces expressed by the co-efficients of the differentials, act on the same particles and have opposite directions in the same lines; wherefore by subtracting the former from the latter, we have,

$$-\frac{d\,.\,\mathrm{V}(r)}{d\,x}\,d\,x-\left(\frac{d\,.\,\mathrm{V}(r)}{d\,y}+f\,y\right)d\,y-\left(\frac{d\,.\,\mathrm{V}(r)}{d\,z}+f\,z\right)d\,z=0, \qquad (8)$$

in which the co-efficients of the differentials express the whole forces urging the molecule in the directions of x, y, z.

It is obvious that the equations (7) and (8) must be identical; for they are both true at every point of the same surface $a\,b\,c$. But if the co-efficients of the differentials of these two equations be identical, the like co-efficients in the equation (5) must be separately equal to zero; and this proves that the co-ordinates of the surface $a\,b\,c$ do not enter into the equation (6), which therefore contains such quantities only as remain invariably the same at all the points of that surface.

The equations (7) and (8) being identical, the latter will belong indifferently to all the similar surfaces in the interior parts, and to the outer surface which is their limit. Wherefore, if for the sake of distinction we suppose that r becomes R at the upper surface, we shall obtain the equation of that surface by integrating, viz.

$$\mathrm{C}=\mathrm{V}(\mathrm{R})+\frac{f}{2}(y^2+z^2). \qquad (9)$$

The integral of (8) will likewise give the equation of any of the interior surfaces, as $a\,b\,c$, viz.

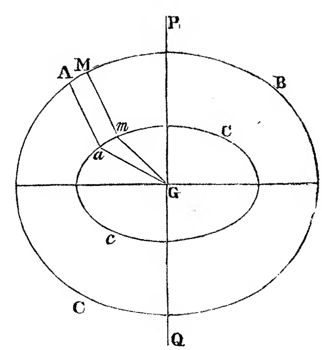
$$p=\mathrm{V}(r)+\frac{f}{2}(y^2+z^2)-\mathrm{C}, \qquad (10)$$

the quantity C being absolutely constant in all circumstances, and the same as in the equation of the upper surface, and p being a new quantity which is constant when the co-ordinates are taken in the surface $a\,b\,c$, but varies when the co-ordinates belong to any point of the mass not contained in that surface. At the upper surface p vanishes; it changes its value in passing from one of the interior surfaces to another; and it is evidently the hydrostatic pressure at every point of the mass, because $-\frac{dp}{dx}$, $-\frac{dp}{dy}$, $-\frac{dp}{dz}$, are equal to the co-efficients of the differentials in equation (8) and to the accelerating forces which oppose and destroy the variation of pressure. The equations (6) and (9) and (10) at which we have arrived by this new train of reasoning are the very same with the equations (4) and (1) and (3) of the first investigation; and as the

remainder of the solution is deduced entirely from these equations, it would be superfluous to repeat here what has already been fully explained. The same procedure as in the first investigation will prove, that the figure of the fluid in equilibrium is exclusively an oblate elliptical spheroid of revolution turning about the less axis h, and that the ratio $\frac{h}{h'}$ of the two axes is derived from the centrifugal force by means of the equation

$$f = B' - \frac{h^2}{h'^2} A'.$$

10. The level surfaces of the mass in equilibrium are properly the interior surfaces similar to the outer surface, and similarly posited about the common centre. Such surfaces agree with CLAIRAUT's definition; for they are perpendicular to the resultant of the forces urging the particles contained in them, as appears from the differential equation (8), which is common to them all. But as every particle within the mass is acted upon by several forces, it may become a question whether there are not other interior surfaces besides those similar to the outer one, which possess the properties of being equably pressed, and of being perpendicular to the resultant of the forces in action. It is this point that we are now to investigate.



Suppose that A B C represents an oblate elliptical spheroid of homogeneous fluid in equilibrium by revolving about the axis P Q; and let $a\,b\,c$ be an oblate elliptical spheroid within A B C, the centres, the less axes, and the equators of the two figures coinciding: taking any particle (x, y, z) of the interior mass $a\,b\,c$, the attractions of the whole mass A B C urging the particle in the respective directions of the co-ordinates, may, as before, be represented by

$$A' x, \qquad B' y, \qquad B' z:$$

and in like manner the attractions of the interior mass $a\,b\,c$ upon the particle, may be denoted by

$$A'' x, \qquad B'' y, \qquad B'' z:$$

and the attractions of the matter between the two surfaces upon the particle, will be

$$(A' - A'')\,x, \qquad (B' - B'')\,y, \qquad B' - B'')\,z.$$

As these forces act upon every particle of the mass $a\,b\,c$, they will cause an internal pressure; let p' denote the hydrostatic pressure at the point (x, y, z) caused by the attraction of the external matter; then, by the general theory, we shall have

$$d\,p' + (A' - A'')\,x\,d\,x + (B' - B'')\,(y\,d\,y + z\,d\,z) = 0;$$

and, by integrating,

$$p' = C' - (A' - A'')\,\frac{x^2}{2} - (B' - B'')\,.\,\frac{y^2 + z^2}{2}. \qquad (11)$$

Further, the joint effect of the centrifugal force and the attraction of the whole mass A B C upon the particle (x, y, z) in the respective directions of the coordinates, is expressed by these forces,

$$A'\,x, \qquad (B' - f)\,y, \qquad (B' - f)\,z:$$

and if p be the pressure thence arising, we shall have

$$d\,p + A'\,x\,d\,x + (B' - f)\,(y\,d\,y + z\,d\,z) = 0;$$

and consequently,

$$p = C - A'\,\frac{x^2}{2} - (B' - f)\,.\,\frac{y^2 + z^2}{2} \qquad (12)$$

which is equivalent to the equation (10), and expresses the whole hydrostatic pressure at every point (x, y, z) within the mass A B C.

In order to form a just notion of the pressures p and p', we shall suppose that the point (x, y, z) is in the interior surface, at $a\,m$: conduct a narrow canal from G to $a\,m$, and continue it outward to the upper surface of the fluid, at A M. Now p is the effort of all the molecules in the canal A $a\,m$ M produced by all the forces that urge them along the canal; and p' is the effort of the canal a G m caused by the attraction of the matter between the two surfaces upon the particles contained in the canal. The pressure p is always directed inward; but the direction in which p' acts will depend upon the na-

ture of the interior spheroid. If it be more oblate than the exterior spheroid A B C, A'' will be greater than A', and the attraction $(A' - A'')\,x$ tending from the equator, the pressure of the canal a G m will be outward and opposed to that of the canal A $a\,m$ M. On this supposition, therefore, the whole action of the matter exterior to the spheroid $a\,b\,c$ will cause a pressure upon the molecule $a\,m$, equal to $p - p'$. By subtracting the equations (11) and (12) we get

$$p - p' = C - C' - A''\frac{x^2}{2} - (B'' - f)\,.\,\frac{y^2 + z^2}{2}: \qquad (13)$$

and we have now to inquire whether a spheroid can be found that will satisfy this equation, on the supposition that $p - p'$ is the same at all the points of the surface of the spheroid.

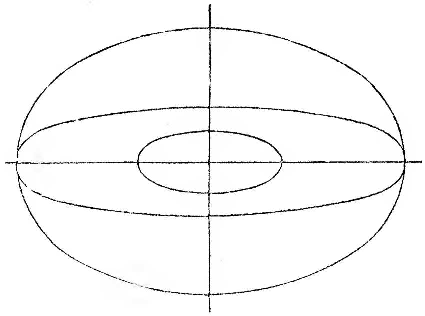
The equation (13) evidently comprehends the level surfaces, which are similar and similarly situated to the upper surface A B C: for, on the supposition that the figures are similar, we have $A' = A''$, $B' = B''$, $p' = C'$, and the equation (13) is identical to the equation (12) which, by giving different values to p, determines all the level surfaces. The equation (13) is similar in its form to the equation (12), A'' and B'' being the same functions of the excentricity of the spheroid $a\,b\,c$, that A' and B' are, of the excentricity of the spheroid A B C; and the centrifugal force f enters alike into both equations. It is therefore evident that the solution of the latter, supposing p constant, and the solution of the former supposing $p - p'$ constant are both contained in the equation,

$$f = B' - \frac{h^2}{h'^2}A':$$

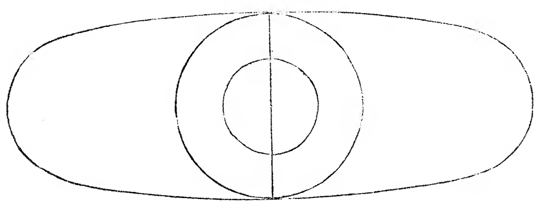
and, as from this two values of $\frac{h^2}{h'^2}$ are in general obtained, one of these results determines the spheroid A B C and its level surfaces, and the other determines the interior spheroid $a\,b\,c$, the surface of which sustains the same pressure at every point by the action of the exterior fluid, and which is therefore separately in equilibrium.

There is this difference between the level surfaces and the other surfaces of equable pressure, that the former spread through the whole mass and ultimately coincide with the upper surface, whereas the latter, on account of the dissimilarity of figure, are confined to a part of the mass. Of the two spheroids

answering to the same centrifugal force, when the exterior one is the less oblate, the greatest interior surface of equable pressure, which is not a level surface, stands upon the equator; and the rest are within this, similar and concentric to it, as in this figure



When the exterior spheroid is the more oblate of the two, the greatest interior surface is described on the less axis, and the rest are similar and concentric to it, as thus,



When the centrifugal force f has a certain relation to the attractive force, the two dissimilar spheroids A B C and a b c coincide in one; and in this case there are no interior surfaces of equable pressure except the level surfaces.

It has now been demonstrated that, in every oblate spheroid in equilibrium by a rotatory motion, there are two sets of interior surfaces equably pressed by the action of the exterior fluid; and, in consequence, that there are two different figures of equilibrium, and only two answering to the same velocity of rotation. But in the hypothesis of the first problem of this paper, and according to the theory of Clairaut, which as far as regards a fluid entirely at liberty, is equivalent to that problem, there is in every case of equilibrium, only one set of interior surfaces equably pressed by the exterior fluid; and this is an incontrovertible proof that the theory of the French geometer is insufficient for determining the figure of equilibrium of a homogeneous planet in a fluid state.

MACLAURIN first demonstrated synthetically the equilibrium of an oblate elliptical spheroid when it revolves about the less axis with a certain angular velocity. In examining the equation of the surface of the fluid, D'ALEMBERT discovered that it admitted of being solved more than one way, that is, he found that there are spheroids of different oblateness which will be in equilibrium with the same velocity of rotation; and LAPLACE proved that there are two such spheroids and no more. Of this truth, first made known merely as a mathematical deduction from an algebraic equation, we have here attempted to give the physical explanation.

Having now fully treated of the equilibrium of a homogeneous fluid, the order of discussion laid down would lead us to investigate that of one of variable density; but the length of this paper makes it advisable to reserve this part of our subject for another occasion.